AN INVESTIGATION OF OPTIMAL SCHEDULING OF UNDERWAY REPLENISHMENTS

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ABSTRACT

The planning for an underway at sea replenishment operation is treated as a special case of the job scheduling problem. Given the service times of the ships to be replenished, general expressions are obtained for total time to complete the operation and total waiting time of the ships involved. Methods to minimize these expressions are developed for several specific cases. Although the largest size of operation considered in this paper is that of three supply ships and three ships to be replenished the methods developed are believed to be extendable to larger operations.

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I. INTRODUCTION TO THE PROBLEM.

One of the techniques most heavily relied upon by the United States Navy in the projection of sea power is that of replenishment at sea. It is this technique which has cut the strings of dependence on land bases which have been tied to combatant ships since steam replaced sail. Because of the assumption of global responsibilities by our nation, and the vulnerability and political aspects of large fixed land bases, it is desirable to make naval task forces as independent of such fixed bases as possible.

Both from a strategic and a tactical standpoint, replenishment at sea is one of the critical operations required of a naval force. While performing this operation, combatant ships are wedded to a relatively slow speed and a steady course. For the ships in any replenishment formation, there can be no immediate high-speed manuvers. The combatants are diverted from their normal operations to a situation which leaves them more vulnerable to every form of attack than is desirable for any length of time. In general, for the duration of the replenishment, the flexibility and responsiveness of the force is greatly degraded.

Replenishment at sea involves two important facets; planning for the operation, in a tactical sense, and the actual conduct of the operation. Considerable effort has been and is being expended to improve the latter. For example, individual ships strive to improve seamanship and loading/unloading plans so as to decrease the time involved in replenishment. New types of supply vessels such as the USS SACRAMENTO (AOE-1) have been built in the past several years, combining the supply functions of two vessels in one ship. More efficient, computer assisted storage and unloading schemes have been instituted on some older and

newer vessels. The use of helicopters to deliver certain items, called vertical replenishment, has been developed, and with the continuing development of higher capacity helicopters such as the UH-46, one can envision the day when only the non-nuclear ships in a task force will be forced to conduct alongside replenishment for fuel (6). Unfortunately, problems such as the present limited strike-down capability of combatants are still to be resolved, and we must live for a while longer with the system we have.

However, no matter how efficient the performance of individual units, the performance of the total operation is strongly dependent on the effectiveness of prior planning of the operation. The planning phase involves detailed knowledge of the capabilities of the units involved, and the interactions involved in the actual conduct of the operation. Certainly, one of the primary concerns of the planning process for current replenishment operations is the answering of the question "what ship goes where, and when?" It is a scheduling problem, and as currently done, is primarily based on experience and common sense.

Analytical studies of the interaction and scheduling aspects of planning for an underway replenishment are few. McCullough (4) has studied the interaction aspect by considering the operation as a queueing process consisting of M service centers in series. The system serves N units, each of which goes through all stages in succession. In order to achieve a steady state solution, the units repeat the process continuously. He then applies the procedure developed by Koenigsberg for cyclic queues to investigate the effect of variation in service rate upon various system characteristics such as mean number of units waiting at a stage and time required for a unit to complete M stages.

The most basic objection to this approach is that it is not a representation of what really happens in a replenishment at sea. First, the cyclic nature of this model is a great simplification of the more complex cycles involved in the operation. Second, the operation is finite, and therefore no steady state exists, forcing one to consider transient conditions. Last, there is the assumption by the author of exponential distributions on the service times to make the model more amenable to analysis. McCullough attempted to resolve some of these objections by developing a simulation program based on his model which more closely approximated reality.

II. OBJECTIVE

An alternative approach which seems worthy of consideration comes from the scheduling area of the problem. This is the tack which we shall pursue in this paper. The objective shall be, in very general terms, to arrive at an analytical method for the determination of the "best" schedule in given circumstances. More specifically, we will analyze the replenishment operation considered as a job scheduling problem where service times are assumed known. For the general case of m supply vessels and n combatants we will attempt to develop methods for determining the optimal solution to the replenishment scheduling problem for two criteria which shall be defined at a later point.

It is appropriate at this point to mention possible bounds on the size of the problem in actual operations. To a fairly close approximation, the largest size situation which might arise is on the order of $m=6$, $n=20$. An operation of this order of magnitude might be expected to occur in Sixth Fleet operations, but, as a matter of actual fact, in the more critical area of operations, the South China Sea, they rarely approach this size. In this area, because of various operational and tactical considerations, the size range should be closer to $1 \geq n \geq 5$, and $1 \geq m \geq 4$.

III. REPLENISHMENT AS A JOB SCHEDULING PROBLEM

If we let each of the m supply ships represent a machine, and each of the n combatants to be serviced represent a job, then our problem would be to find the optimal schedule for the n jobs on the m machines. The nature of underway replenishment introduces a unique twist to the job scheduling problem; that is the fact that several jobs may be started at once by different machines. A literature search revealed a considerable body of work done in the area of job scheduling[1], but none which was applicable to the problem with this twist. All of the work was concerned with the case in which all jobs must start at the first machine. Such a constraint would seldom, if ever be associated with replenishment at sea. Unfortunately, the removal of this constraint means that we must seek a general solution to a more complex combinatorial problem than the "one line" case previously studied in the literature.

Elaborating on the notion of replenishment as a job scheduling problem, let us assume that we know the service time of each combatant at each supply ship, and that transit times between supply ships are identical for all combatants and negligible in comparison with service times. Further, let us assume that for this travel between supply ships, there exists a specified flow, say, from left to right for example. This means that the flow of jobs through the system can be represented as

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[1]See, for example, Dudek and Teuton (1), Eastman, Even and Isaacs (2), Lomnicki (3), Palmer (5), and Smith and Teuton (7).

indicated in the figure below for the case m=3:

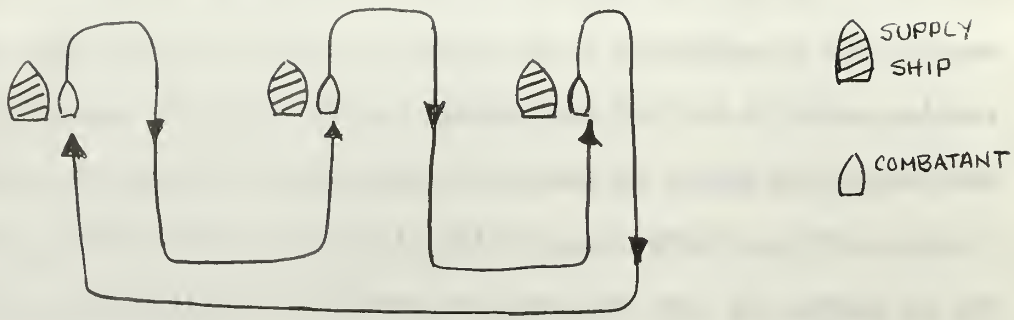

Figure 1

For the moment we will assume that we have not specified a particular machine (supply ship) ordering.

One may object that by assuming a specified flow pattern, we have deleted many possible solutions to the problem. While this is true, it is important to realize that the physical conditions under which a replenishment at sea operates preclude a large number of such possible flow patterns for reasons involving safety or good seamanship. It is a violation of the above principle to maneuver in the manner which would be required for certain flow patterns in the presence of the replenishment formation, which is a privileged formation. However, by assuming no specification of supply ship order, we do allow a larger number of patterns to be considered than appears feasible at first glance.

The only feasible case not allowed by our formulation of the system is that which we will call "jockeying", described below for the case m=3:

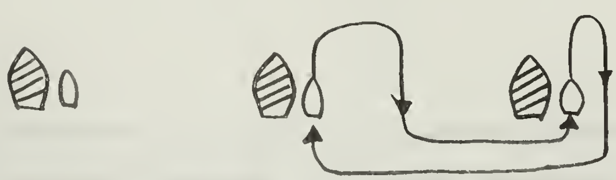

Figure 2

Suppose that, as depicted in Figure 2, the combatant on the left takes much longer than the right hand combination to complete its service; that is, the two combatants at the right may find themselves waiting for the left-hand combatant to finish its service. Then, the case may arise in which one would want to interchange the two rather than sending one or both to wait behind the left-most station. Note that the same situation may apply to any of the combinations of two stations in the system. For larger systems, the same type of situation may occur, but will be more complex.

IV. OPTIMALITY CRITERIA.

There are two optimality criteria which appear appropriate for our problem. The first is the minimum total time to complete the entire replenishment operation, or "Min T" criterion. There is some schedule or set of schedules which will yield this Min T. This particular criterion may be the desirable path to follow when the operation is to be performed under a threat of some sort; air, surface, or sub-surface.

The second criterion is less obvious. One might wish to minimize the total waiting time of all ships concerned, both combatant and supply. By definition, and in consonance with reality, a ship is not considered to be waiting once she has completed all her required services. This criterion might be applied where one is concerned more about the efficiency and smoothness of the operation than about total time to complete. We will call this the "Min M" criterion. This criterion will give total replenishment times which are greater than or equal to the value of Min T.

It is important to note that careful consideration of the tactical situation will be necessary in making a choice between the criterion. For example, a situation may arise under threat for which it will be more desirable to use the Min M criterion to disperse the combatant force more rapidly, then to use the Min T criterion, which may keep a large number of combatants near the supply ships throughout the operation.

V. TOTAL WAITING AND COMPLETION TIMES

A. Matrix of Service Times

Let us assume there are m supply ships and n combatants participating in the replenishment operation. Let the supply ships be identified by the numbers one through m and the combatants by the numbers one through n. By letting S_{ij} represent the known service time of combatant i at supply ship j, the nm services may be expressed as a matrix $\mathcal{S}$. For example, in the case n = m = 3,

$$\mathcal{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

The i^{th} row of $\mathcal{S}$ is the vector of service times for combatant i and the j column of $\mathcal{S}$ is the service vector of supply ship j. As defined this matrix of service times is dependent upon the indexing procedure used to number the combatant and supply ships. Since there are n!m! ways in which the n combatants and m supply ships may be numbered there are necessarily n!m! unique matrices which could be constructed from the nm service times. The relationship between the n!m! matrices that could be obtained is that given any one we may generate the remaining by elementary row and column transformations of the given matrix. In other words, there are n!m! ways in which the rows and columns of $\mathcal{S}$ may be arranged and each of these corresponds to one way of numbering the n combatants and m supply ships. The dependence of $\mathcal{S}$ upon a particular numbering system can be removed by relaxing the usual definition for equality between matrices. For the purpose of this paper two service time matrices are <u>equal</u> provided they are row and/or column equivalent.

B. The Solution Matrix

The flow or movement of combatants from one supply ship to the next has been assumed to be ordered and so a solution is complete if we merely specify the starting order. This specification will take the form of assigning stations to the m supply ships and starting positions to the n combatants. For the purpose of analysis no generality is lost by visualizing the m supply stations as being abreast and numbering the stations from left to right with the numbers one through m. When n = m the m combatant starting positions are taken to be those stations which initially are identical to the m supply stations.

The question of combatant starting positions is not so easily handled when n is not equal to m. Such a case may be dealt with by creating either false supply ships or false combatants depending on whether n is greater than or less than m. A false ship of either type will have for its service vector a vector of zeroes. The use of false ships reduces all cases to the form m = n. To make this notion clear the case n = 3, m = 2 can be visualized as depicted in Figure 3.

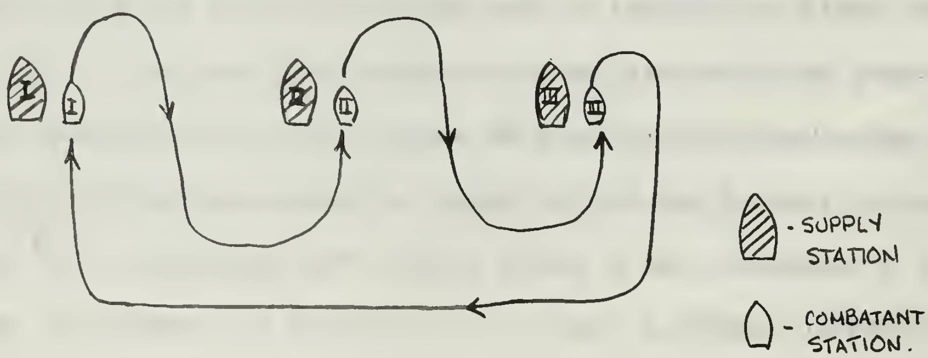

Figure 3

In this case one of the three supply stations is actually filled with a false supply ship which means that the combatant assigned to that station will be initially waiting for service. A combatant station

having once been assigned moves with the particular combatant assigned to that station. In the example above the positions after completion of the first services would be as given in Figure 4.

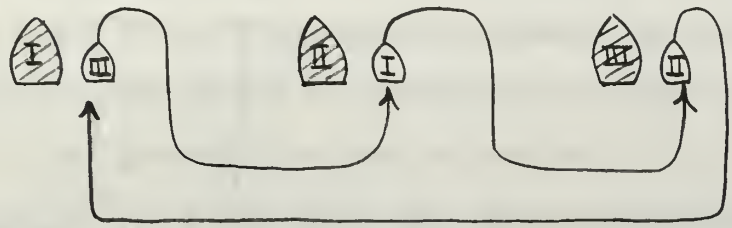

Figure 4

Returning now to the matrix $\mathcal{S}$, the elements S_{ij} may be given a more specific meaning by the use of superscripts which represent their positions within the matrix and denote the assignment of stations. The element S_{ij}^{KL} is thus the service time of combatant i at supply ship j and is also combatant station K at supply station L. With superscripts attached to its elements, each one of the $(m!)^2$ ways that $\mathcal{S}$ may be expressed becomes a solution to the scheduling problem. The main diagonal of a superscripted $\mathcal{S}$ matrix is the first set of services to be performed and this specification constitutes a solution. To determine an optimum solution, a rule must be devised that will indicate how to arrange the rows and columns of $\mathcal{S}$ so that when it is viewed as a solution matrix either total system completion time or total waiting time is a minimum.

In devising such a rule it is convenient to associate a matrix A with the position of the elements S_{ij}^{KL}. That is $a_{KL} = S_{ij}^{KL}$. We will work henceforth with the position matrix A but it is important to understand that when we speak of minimizing or maximizing a function of the a_{ij}'s, we are actually referring to the rearrangement of the rows and columns of $\mathcal{S}$ so as to achieve such a minimum or maximum.

C. Duplicate Solutions

The matrix $\mathcal{S}$ is unfortunately repetitious in solutions which can be obtained by the rearrangement of its rows and columns. For example if the rows and columns of the matrix $\begin{bmatrix} 5 & 2 & 7 \\ 3 & 3 & 4 \\ 1 & 2 & 4 \end{bmatrix}$ are rearranged to either of the following forms: $\begin{bmatrix} 4 & 1 & 2 \\ 7 & 5 & 2 \\ 4 & 3 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 4 & 3 \\ 2 & 4 & 1 \\ 2 & 7 & 5 \end{bmatrix}$ no new information is obtained. The three matrices are identical solutions under our previous definitions. In the general case there are $(m!)^2$ orderings of the rows and columns of $\mathcal{S}$, but $(m!)^2\ (1-1/m)$ of these solutions are duplicates leaving only $m!(m-1)!$ <u>different</u> solutions. To avoid such duplication we will always assign the supply ship having $\underset{j}{\text{MAX}} \sum_i S_{ij}$ to supply station number 1. Such a convention does not further restrict our solution.

D. Total Waiting Time

The following notation will be used in the development of expressions for the waiting and completion times. If the operation <u>starts at time zero</u>, then:

T = total system time,

τ_i = the time that the combatant assigned to station i completes its final service,

t_i = the time that the supply ship assigned to station i completes its final service,

$a_{i\cdot} = \sum_j a_{ij}$ = the total of all service times for the combatant assigned to station i,

$a._j = \sum_i a_{ij}$ = the total of all service times for the supply ship assigned to station j,

w_{ij} = waiting time of the combatant assigned to station i for service at the supply ship assigned to station j.

W_{ij} = waiting time of the supply ship assigned to station j for the combatant assigned to station i,

$w_{i.} = \sum_j w_{ij}$ total waiting time for the combatant assigned to station i,

$W_j = \sum_i W_{ij}$ = total waiting time for the supply ship assigned to station j,

$w = \sum_i w_i$ = total waiting time of the combatants,

$W = \sum_j W_j$ = total waiting time of the supply ships.

Under the ordered flow which has been specified, several relations are immediately obvious:

$$(1) \quad t_i = \begin{cases} \tau_i + 1 & i \neq n \\ \tau_1 & i = n \end{cases}$$

(2) (a) $W_i = t_i - a.i$

(b) $W_i = \tau_i - a_i.$

Also apparent is the fact that W = w, since:

$$W = \sum_i W_i = \sum_i (t_i - a.i) = \sum_i t_i - \sum_i a.i = \tau_1 + \sum_{i=2}^{n} \tau_i - \sum_i a_{i.} = \sum_i \tau_i - \sum_i a_{i.}$$

$$= w$$

The general expressions for W_{ij} and w_{ij} are:

$$(3) \quad W_{ij} = \begin{cases} \text{MAX}\left[a_{i+1j} + w_{i+1j} - a_{ij-1} - W_{ij-1}, 0\right] & i \neq j \\ 0 & i = J \end{cases}$$

$$(4)\quad W_{ij} = \begin{cases} \text{MAX}\left[a_{ij-1} + w_{ij-1} - a_{i+1j} - w_{i+1j}, 0\right] & i \neq j \\ 0 & i = j \end{cases}$$

where the subscript $0 \equiv n$ and the subscript $n + 1 \equiv 1$.

Now w and W may be written as,

$$w = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} = \sum_{i=0}^{n-1}\left[\sum_{j=1}^{n-i} w_{ji+j} + \sum_{j=n-i+1}^{n} w_{jj-n+i}\right] = \sum_{i=1}^{n-1}\left[\sum_{j=1}^{n-i} w_{ji+j} + \sum_{j=n-i+1}^{n} w_{jj-n+i}\right]$$

$$W = \sum_{i=1}^{n}\sum_{j=1}^{n} W_{ij} = \sum_{i=0}^{n-1}\left[\sum_{j=1}^{n-i} W_{ji+j} + \sum_{j=n-i+1}^{n} W_{jj-n+i}\right] = \sum_{i=1}^{n-1}\left[\sum_{j=1}^{n-i} W_{ji+j} + \sum_{j=n-i+1}^{n} W_{jj-n+i}\right]$$

If we let $M = W + w$ then,

$$M = \sum_{i=1}^{n-1}\left[\sum_{j=1}^{n-i}(W_{ji+j} + w_{ji+j}) + \sum_{j=n-i+1}^{n}(W_{jj-n+i} + w_{jj-n+i})\right]$$

and using equations (3) and (4),

$$M = \sum_{i=1}^{n-1}\left\{\sum_{j=1}^{n-i}\left(\text{MAX}\left[a_{j+1i+j}+w_{j+1i+j}-a_{ji+j-1}-W_{ji+j-1},0\right] + \text{MAX}\left[a_{ji+j-1} + W_{ji+j-1}-a_{j+1i+j}-w_{j+1i+j},0\right]\right)\right.$$

$$\left. + \sum_{j=n-i+1}^{n}\left(\text{MAX}\left[a_{j+1j-n+i}+w_{j+1j-n+i}-a_{jj-n+i-1}-W_{jj-n+i-1},0\right] + \text{MAX}\left[a_{jj-n+i-1}+W_{jj-n+i-1} - a_{j+1j-n+i} - w_{j+1j-n+i}, 0\right]\right)\right\}$$

Finally, since MAX $(A-B,0)$ + MAX $(B-A,0) = |A-B|$,

$$(5)\quad M = \sum_{i=1}^{n-1}\left[\sum_{j=1}^{n-i}\left|a_{j+1i+j}+w_{j+1i+j}-a_{ji+j-1}-W_{ji+j-1}\right| + \sum_{j=n-i+1}^{n}\left|a_{j+1j-n+i} + w_{j+1j-n+i}-a_{jj-n+i-1}-W_{jj-n+i-1}\right|\right]$$

To minimize the waiting time within the system we have only to minimize M which is always twice the total waiting time.

This fact is a direct result of the preceding work but can be justifiably questioned on the grounds of its interpretation. The use

of false ships necessarily causes their waiting times to be contained in M and these waiting times are, in a practical sense, of no concern to us. We are interested instead in the total waiting time of the ships that actually exist, the <u>real</u> total waiting time. Since we must either create false supply ships or false combatant ships, but never both, there is a simple relationship between total waiting time and real total waiting time. The quantity M/2, in terms of real waiting times, may be summarized as:

$$M/2 = \begin{cases} \text{The real total waiting time of the combatants} & \text{if false supply ships were created,} \\ \text{The real total waiting time of the supply ships} & \text{if false combatants were created,} \\ \text{The real total waiting time of the combatants or supply ships} & \text{if no false ships were created} \end{cases}$$

Notice that if a transit time, identical for all combatants, is added to each a_{ij} these constants would cancel in equation (5). We can thus ignore transit times provided they are assumed to be identical for all combatants.

E. <u>Total Completion Time</u>

It is apparent from equation (1) that the total system time T is given by,

$$T = \text{MAX } (\tau_1, \tau_2, \tau_3 \ldots \tau_n) = \text{MAX } (t_1, t_2, t_3 \ldots\ldots t_n).$$

Or from equation (2), (a) and (b),

$$T = \text{MAX } (w_1+a_{1.}, w_2+a_{2.} \ldots\ldots w_n+a_{n.}) = \text{MAX } (W_1+a_{.1}, \ldots\ldots W_n+a_{.n}).$$

$$(6)\quad T = \mathrm{MAX}\begin{bmatrix} \sum_j w_{1j}+a_{1.} \\ \sum_j w_{2j}+a_{21} \\ \vdots \\ \sum_j w_{nj}+a_{n.} \end{bmatrix} = \mathrm{MAX}\begin{bmatrix} \sum_i W_{i1}+a_{.1} \\ \sum_i W_{i2}+a_{.2} \\ \vdots \\ \sum_i W_{in}+a_{.n} \end{bmatrix}$$

Having developed general expressions for M and T we will now turn our attention to their minimization.

VI. OPTIMAL SOLUTIONS

A. Solutions for General m and n

We have not as yet been successful in obtaining general solutions for the required minimizations of M and T. Our efforts to date have been directed toward solving specifie cases, i.e., given values of n, with the hope of eventually providing an inductive argument for the general case. Although it is unfortunate that no general solution has yet been found there is some consolation in the fact that the cases of small m and n are the ones that most often occur in the real world. Since the relevant cases are those of small m and n, optimal solutions for these few cases may be all that will ever be required.

B. Solutions for Specific Cases

The remainder of this report is concerned with the beginning of what we eventually hope will be a compilation of optimal solutions for the relevant cases. In particular, solutions are presented for MIN M and MIN T in the cases n = 2 and n = 3. These solutions are in the immediately following sections. Section 5 presents a methodology for obtaining MIN M by bounds. Although the procedure outlined is that for the case n = 3, we believe that the principle value of the method will come from its extension to larger cases.

1. MIN M in the Case n = 2

To minimize M we attempt to find some function of the a_{ij}'s such that M in terms of this function will indicate how the rows and columns of $\mathcal{S}$ should be arranged. In order to clarify this technique we will explore at some length the simple case n = 2. For this case equation (5) becomes:

$$M = |a_{22} - a_{11}| + |a_{11} - a_{22}| = 2\,|a_{11} - a_{22}|\,.$$

First we note the obvious fact that the row and column sums of are arranged according to the way the rows and columns of $\mathcal{S}$ are arranged. This fact suggests that if M is expressed in terms of row and column sums and if this expression indicates how to arrange the row and column sums of $\mathcal{S}$ so as to minimize M, then we will also know how to arrange the rows and columns of $\mathcal{S}$ so as to minimize M. In terms of row and column sums,

$$M = 2 \left| a_{.1} - a_{2.} \right| .$$

Since

$$a_{.1} = \text{MAX } (S_{.1}, S_{.2}),$$

in keeping with section C, it is obvious that to minimize M,

$$a_{2.} = \text{MAX } (S_{1.}, S_{2.}).$$

We are told how to arrange the row and column sums of $\mathcal{S}$ and, therefore, how to arrange the rows and columns of $\mathcal{S}$.

A less obvious solution procedure is also worthy of consideration. We define

$$A_{ij} = a_{i.} - a_{ij} , \bar{A}_{ij} = a_{.j} - a_{ij}$$

and construct the following matrix:

$$\bar{\bar{A}} = \begin{bmatrix} A_{11}, \bar{A}_{11} & A_{12}, \bar{A}_{12} \\ A_{21}, \bar{A}_{21} & A_{22}, \bar{A}_{22} \end{bmatrix}$$

In these terms,

$$M = 2 \left| A_{21} - \bar{A}_{21} \right| ,$$

which again suggests a method of arranging the rows of $\mathcal{S}$.

Consider the following example:

$$\mathcal{S} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \qquad \bar{\bar{\mathcal{S}}} = \begin{bmatrix} 3,2 & 4,1 \\ 4,1 & 2,3 \end{bmatrix}$$

Since M is only a function of the pair of numbers that appear as the first element in the second row (the 21 position) of the $\bar{\bar{\mathcal{S}}}$ matrix,

it is apparent that to minimize M we must interchange the first and second row of $\bar{\bar{\mathcal{S}}}$ which corresponds identically with interchanging the first and second row of $\mathcal{S}$.

The number of other functions of the a_{ij}'s which could be used to solve for the minimum of M becomes larger as n increases. The problem is to find one for each case, or eventually, one for all cases which will lead to the optimum arrangement of the rows and columns of $\mathcal{S}$.

2. <u>MIN M for the Case n = 3</u>

To obtain an expression for M in the case n = 3, equation (5) is expanded using the relation MAX (A-B,0) = MAX (A,B)-B. This gives:

$$(7)\quad M = |a_{11}-a_{22}| + |a_{22}-a_{33}| + |a_{33}-a_{11}|$$
$$+|a_{23}+\mathrm{MAX}(a_{22},a_{33})-a_{12}-\mathrm{MAX}(a_{11},a_{22})|$$
$$+|a_{31}+\mathrm{MAX}(a_{11},a_{33})-a_{23}-\mathrm{MAX}(a_{22},a_{33})|$$
$$+|a_{12}+\mathrm{MAX}(a_{11},a_{22})-a_{31}-\mathrm{MAX}(a_{11},a_{33})|$$

Equation (7) may be reduced using the relation $|A-B| = 2\mathrm{MAX}(A,B)-A-B$ and the definitions of A_{ij} and $\bar{A}_{ij}$ to:

$$(8)\quad M/2 = \mathrm{MAX}(A_{13},\bar{A}_{13},A_{21},\bar{A}_{32}) + \mathrm{MAX}(A_{21},\bar{A}_{21},A_{32},\bar{A}_{13})$$
$$+ \mathrm{MAX}(A_{32},\bar{A}_{32},A_{13},\bar{A}_{21}) - (A_{13}+A_{21}+A_{32}).$$

This equation shows that M/2 depends only on the elements that appear in the 13, 21 and 32 positions of the $\bar{\bar{\mathcal{S}}}$ matrix. By the rearrangement of the rows and columns of $\bar{\bar{\mathcal{S}}}$ there are twelve ways that these positions can be filled. It will be noted, however, that for any three given elements there are two ways in which they can fill the three positions. This observation permits us to divide the search for an optimum solution into two parts. We first identify the three elements from $\bar{\bar{\mathcal{S}}}$ which give rise to the optimum solution and then determine how these three elements are to be arranged to fill the 13, 21 and 32 positions of the matrix $\bar{\bar{A}}$.

To see how the second part is to be done, assume first that the three elements that minimize M/2 are given from the rows of $\bar{\bar{\mathcal{S}}}$. Let these be:

$\bar{\bar{A}}_{13} = 8,5$ $\qquad\qquad$ $\bar{\bar{A}}_{21} = 4,7$ $\qquad\qquad$ $\bar{\bar{A}}_{32} = 6,6$

From equation (8) M/2 = 16 + 7 - 18 = 5 for this arrangement of the given elements. It is possible, however, by a rearrangement of the rows and columns of $\bar{\bar{\mathcal{S}}}$ to obtain a second solution using these elements. That is:

$\bar{\bar{A}}_{13} = 6,6$ $\qquad\qquad$ $\bar{\bar{A}}_{21} = 4,7$ $\qquad\qquad$ $\bar{\bar{A}}_{32} = 8,5$

For this arrangement M/2 = 16 + 6 - 18 = 4 which is the optimum solution. To determine which set of elements gives rise to the optimum solution requires an inspection of each of the six choices available. For convenience the six choices may be expressed as the rows and columns of a matrix:

$$\bar{\bar{\bar{\mathcal{S}}}} \equiv \begin{bmatrix} \bar{\bar{s}}_{11} & \bar{\bar{s}}_{23} & \bar{\bar{s}}_{32} \\ \bar{\bar{s}}_{22} & \bar{\bar{s}}_{31} & \bar{\bar{s}}_{13} \\ \bar{\bar{s}}_{33} & \bar{\bar{s}}_{12} & \bar{\bar{s}}_{21} \end{bmatrix}$$

Since $A_{13}+A_{21}+A_{32} = \bar{A}_{13}+\bar{A}_{21}+\bar{A}_{32}$ it may be verified from equation (8) that the minimum value for M/2 for each row or column of $\bar{\bar{\bar{\mathcal{S}}}}$ is determined by the following set of rules.

1) If the $\text{MAX}\left[A_{13}, A_{21}, A_{32}, \bar{A}_{13}, \bar{A}_{21}, \bar{A}_{32}\right] = A_{ij}$,

then,

$$M/2 = A_{ij} + \text{MAX}\left[A_{kl}, A_{nm}, \bar{A}_{ij}, \text{MIN}(\bar{A}_{kl}, \bar{A}_{nm}) \right] - A_{kl} - A_{nm}$$

$$k \neq i \neq n$$

$$l \neq j \neq m$$

2) If the $\text{MAX}\left[A_{13}, A_{21}, A_{32}, \bar{A}_{13}, \bar{A}_{21}, \bar{A}_{32}\right] = \bar{A}_{ij}$,

then,

$$M/2 = \bar{A}_{ij} + \text{MAX}\left[\bar{A}_{kl}, \bar{A}_{nm}, A_{ij}, \text{MIN}(A_{kl}, A_{nm}) \right] - \bar{A}_{kl} - \bar{A}_{nm}$$

$$k \neq i \neq n$$

$$l \neq j \neq m$$

A detailed example should serve to make the method clear.

Let:

$$\mathcal{S} = \begin{bmatrix} 4 & 6 & 3 \\ 7 & 2 & 5 \\ 4 & 2 & 1 \end{bmatrix}$$

Then:

$$\bar{\bar{\mathcal{S}}} = \begin{bmatrix} 9,11 & 7,4 & 10,6 \\ 7,8 & 12,8 & 9,4 \\ 3,11 & 5,8 & 6,8 \end{bmatrix}$$

and:

$$\bar{\bar{\bar{\mathcal{S}}}} = \begin{bmatrix} 9,11 & 9,4 & 5,8 \\ 12,8 & 3,11 & 10,6 \\ 6,8 & 7,4 & 7,8 \end{bmatrix}$$

From the first column of $\bar{\bar{\bar{\mathcal{S}}}}$, $M/2 = 12 + \mathrm{MAX}\left[9,6,8,\mathrm{MIN}(11,8)\right]$

$$- 9 - 6 = 6.$$

By similar calculations the following results are obtained:

$$\bar{\bar{\bar{\mathcal{S}}}} = \begin{array}{c c} \begin{bmatrix} 9,11 & 9,4 & 5,8 \\ 12,8 & 3,11 & 10,6 \\ 6,8 & 7,4 & 7,8 \end{bmatrix} & \begin{array}{l} M/2 = 8 \\ M/2 = 9 \\ M/2 = 4 \end{array} \\ \begin{array}{ccc} M/2=6 & M/2=10 & M/2=6 \end{array} & \end{array}$$

Since the third row of $\bar{\bar{\bar{\mathcal{S}}}}$ gives the minimum value for M/2, the elements which minimize M/2 are $\{6,8 \quad 7,4 \quad 7,8\}$. From $\bar{\bar{\mathcal{S}}}$ it is apparent that by interchanging the second and third columns we will have:

$$\bar{\bar{A}}_{21} = 7,8 \qquad \bar{\bar{A}}_{32} = 6,8 \quad \text{and } \bar{\bar{A}}_{13} = 7,4$$

From equation (8) this arrangement gives:

$$\frac{M}{2} = 8 + 8 + 8 - 20 = 4$$

By appropriately interchanging the rows and columns of $\mathcal{S}$ we may also obtain:

$$\bar{\bar{A}}_{21} = 7,8 \qquad \bar{\bar{A}}_{32} = 7,4 \quad \text{and } \bar{\bar{A}}_{13} = 6,8$$

which gives M/2 = 4. There are thus two optimum solutions to the given problem. The first is obtained by interchanging the second and third

column of $\mathcal{S}$ and the second by interchanging the first and third rows of $\mathcal{S}$. The optimal solutions are:

$$\mathcal{S}_1 = \begin{bmatrix} 4 & 3 & 6 \\ 7 & 5 & 2 \\ 4 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathcal{S}_2 = \begin{bmatrix} 4 & 2 & 1 \\ 7 & 2 & 5 \\ 4 & 6 & 3 \end{bmatrix}$$

For each solution the waiting time of the combatants is 4 and the waiting time of the supply ships is 4 since no false ships were created.

3. MIN T for the Case n = 2

For the case n = 2, equation (6) gives:

$$T = \text{MAX} \begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ a_{\cdot 1} \end{bmatrix}$$

Since

$$\text{MIN}\left(\text{MAX} \begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ a_{\cdot 1} \end{bmatrix}\right) = \text{MAX} \begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ a_{\cdot 1} \end{bmatrix}$$

<u>any</u> <u>arrangement</u> of the rows and columns of $\mathcal{S}$ is an optimum solution for T in the case n = 2 .

4. MIN T for the Case n = 3

From equation (6), T for the case n = 3 becomes, after some reduction,

$$T = \text{MAX} \begin{bmatrix} \text{MAX}\left[a_{23} + \text{MAX}(a_{22}, a_{33}),\ a_{12} + \text{MAX}(a_{11}, a_{22})\right] + a_{13} \\ \text{MAX}\left[a_{31} + \text{MAX}(a_{11}, a_{33}),\ a_{23} + \text{MAX}(a_{22}, a_{33})\right] + a_{21} \\ \text{MAX}\left[a_{12} + \text{MAX}(a_{11}, a_{22}),\ a_{31} + \text{MAX}(a_{11}, a_{33})\right] + a_{32} \end{bmatrix}$$

which may be expanded to:

$$T = \text{MAX}\begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ a_{3\cdot} \\ a_{\cdot 1} \\ a_{11} + a_{12} + a_{32} \\ a_{11} + a_{31} + a_{32} \\ a_{12} + a_{13} + a_{22} \\ a_{13} + a_{22} + a_{23} \\ a_{21} + a_{23} + a_{33} \\ a_{21} + a_{31} + a_{33} \end{bmatrix}$$

Because in this equation $\text{MAX}\begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ a_{3\cdot} \\ a_{\cdot 1} \end{bmatrix}$ cannot be altered for any given problem, every rearrangement of $\mathcal{S}$ will give the same value for this expression. This fact suggests that an alternative approach to seeking a solution having MIN T is to seek a solution giving a value of $\mathcal{Z} \leq 0$ where,

$$(9) \quad \mathcal{Z} = \text{MAX}\begin{bmatrix} a_{11} + a_{12} + a_{32} \\ a_{11} + a_{31} + a_{32} \\ a_{12} + a_{13} + a_{22} \\ a_{13} + a_{22} + a_{23} \\ a_{21} + a_{23} + a_{33} \\ a_{21} + a_{31} + a_{33} \end{bmatrix} - \text{MAX}\begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ a_{3\cdot} \\ a_{\cdot 1} \end{bmatrix} .$$

Any arrangement of $\mathcal{S}$ which yields a value of $\mathcal{Z} \leq 0$ is an optimal solution for the minimization of T and therefore we need not seek that

particular solution giving the MIN $\mathcal{Z}$ value. There could be perhaps a number of solutions for which $\mathcal{Z} \leq 0$. This may be interpreted to mean that within the minimization of T there may exist some latitude for the adjustment of M. In equation (9) it may again be observed that the inclusion of transit times for combatants would result in their cancellation if all transit times are equal. An alternative form of equation (9) is given by equation (10).

$$(10) \quad \mathcal{Z} + \text{MAX}\begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ a_{3\cdot} \\ a_{\cdot 1} \end{bmatrix} = \text{MAX}\begin{bmatrix} a_{12} + \text{MAX}\begin{pmatrix} a_{13} + a_{22} \\ a_{11} + a_{32} \end{pmatrix} \\ a_{23} + \text{MAX}\begin{pmatrix} a_{13} + a_{22} \\ a_{21} + a_{33} \end{pmatrix} \\ a_{31} + \text{MAX}\begin{pmatrix} a_{11} + a_{32} \\ a_{21} + a_{33} \end{pmatrix} \end{bmatrix}$$

For an arrangement of $\mathcal{S}$ to be an optimum solution for MIN T the right hand side of equation (10) must be less than or equal to MAX $\begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ a_{3\cdot} \\ a_{\cdot 1} \end{bmatrix}$.

The right hand side of (10) shows that the elements which appear in the 12, 23 and 31 positions of the $\mathcal{S}$ matrix are the ones, subject to additive terms, that concern us. The term to be added to one of these position elements results from a matching of the elements in its row with those in its column. For the term $a_{12} + \text{MAX}\begin{pmatrix} a_{13} + a_{22} \\ a_{11} + a_{32} \end{pmatrix}$, Figure 5 shows the matching which pertains.

$$\begin{bmatrix} a_{11} & \boxed{a_{12}} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Figure 5

For a given matrix $\mathcal{S}'$ there are twelve ways that the 12, 23 and 31 positions may be filled. It may be shown, however, that T is only dependent on those numbers which fill these positions and not on the order in which they are filled. This means that there are at most six different T solutions (3!). For a <u>given arrangement</u> of the matrix $\mathcal{S}'$, the six different choices to fill the three positions may be expressed as the rows and columns of a matrix A'. The superscripts on the elements of A' denote only matrix position.

$$A' = \begin{bmatrix} a^{11}_{21} & a^{12}_{32} & a^{13}_{13} \\ a^{21}_{33} & a^{22}_{11} & a^{23}_{22} \\ a^{31}_{12} & a^{32}_{23} & a^{33}_{31} \end{bmatrix}$$

The values of $\mathcal{Z}$ for the six different choices of the elements to fill the 12, 23 and 31 positions of $\mathcal{S}'$, may be obtained by the use of the following equations.

For the i^{th} row of A':

$$(11) \quad \mathcal{Z} = \underset{j}{\text{MAX}} \left[a^{ij} + \underset{k}{\text{MAX}} \left(\sum_{l \neq i} a^{lk} \right) \right] - \text{MAX} \begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ a_{3\cdot} \\ a_{\cdot 1} \end{bmatrix} .$$

For the j^{th} column of A':

$$(12)\quad Z = \underset{i}{\mathrm{MAX}} \left[a^{ij} + \underset{L}{\mathrm{MAX}} \left(\sum_{K \neq j} a^{Lk} \right) \right. - \mathrm{MAX} \begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ a_{3\cdot} \\ a_{\cdot 1} \end{bmatrix}.$$

An example will show the procedure outlined above to be quite simple to apply. Consider the following matrix:

$$S = \begin{bmatrix} 8 & 3 & 5 \\ 2 & 6 & 4 \\ 7 & 5 & 3 \end{bmatrix}$$

The associated A' matrix for this arrangement of S is

$$A' = \begin{bmatrix} 2 & 5 & 5 \\ 3 & 8 & 6 \\ 3 & 4 & 7 \end{bmatrix}$$

Since the $\mathrm{MAX} \begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ a_{3\cdot} \\ a_{\cdot 1} \end{bmatrix} = 17$, equation (11) for the first row of A' gives

$$Z = \mathrm{MAX} \left[2+\mathrm{MAX} \begin{pmatrix} 8+4 \\ 6+7 \end{pmatrix},\ 5+\mathrm{MAX} \begin{pmatrix} 3+3 \\ 6+7 \end{pmatrix},\ 5+\mathrm{MAX} \begin{pmatrix} 3+3 \\ 8+4 \end{pmatrix} \right] - 17$$

$$= \mathrm{MAX} \left[15,\ 18,\ 17 \right] - 17 = 1$$

With equations (11) and (12) in mind the remaining cases may be examined without resorting to written form. Since $\mathrm{MAX} \begin{bmatrix} a_{1\cdot} \\ a_{2\cdot} \\ a_{3\cdot} \\ a_{\cdot 1} \end{bmatrix}$ is the

same for each solution it is only necessary to scan a possible solution for combinations which disqualify it from giving a minimum value for T. For example, the second row of A' is disqualified by the combination 8+5+7 = 20. By this procedure the only solution not disqualified is the first column of A' which gives $\mathcal{Z} = 0$. The elements of the first rwo of A' correspond to the elements in the 21, 33 and 12 positions of the matrix $\mathcal{S}$.

There are two ways to rearrange $\mathcal{S}$ so that these elements appear in the 21, 32 and 13 positions. The two solutions are:

$$\mathcal{S}_1 = \begin{bmatrix} 8 & 5 & 3 \\ 2 & 4 & 6 \\ 7 & 3 & 5 \end{bmatrix} \quad \mathcal{S}_2 = \begin{bmatrix} 7 & 5 & 3 \\ 2 & 6 & 4 \\ 8 & 3 & 5 \end{bmatrix}$$

Each of these solutions gives a value of T = 17 which is the minimum possible.

There do exist cases for which MIN $\mathcal{Z} > 0$. These become obvious when it turns out that all possible solutions have been disqualified. In such a case it is only necessary to repeat the solution processes eliminating a particular solution only after a better solution is found. An example is:

$$\mathcal{S} = \begin{bmatrix} 4 & 1 & 6 \\ 3 & 2 & 4 \\ 5 & 6 & 1 \end{bmatrix}$$

for which

$$A' = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 4 & 2 \\ 1 & 4 & 5 \end{bmatrix}$$

It is easily verified that MIN $\mathcal{Z} = 1$ and that there are four minimum solutions.

5. MIN M by Bounds for the Case n = 3

There are twelve feasible solutions to our problem in the case n = 3. If upper and lower bounds can be placed on the value of M for these solutions then perhaps the set of solutions we must consider in detail may be a smaller set than the original twelve. We begin by determining the expressions for the upper and lower bounds on the value of M for a particular solution.

From equations (1) and (2),

$$w_i+a_{i\cdot} = \begin{cases} W_{i-1}+a_{\cdot i-1} & i\neq 1 \\ W_n+a_{\cdot n} & i=1 \end{cases},$$

which can be written as:

$$|w_i-W_{i-1}| = \begin{cases} |a_{\cdot i-1}-a_{i\cdot}| & i\neq 1 \\ |a_{\cdot n}-a_{1\cdot}| & i=1 \end{cases}$$

The latter relation leads to a lower bound for M. Since there cannot exist a difference between waiting times unless waiting time exists, we can state that

$$M \geqslant \sum_{i=2}^{n} |a_{\cdot i-1}-a_{i\cdot}| + |a_{\cdot n}-a_{1\cdot}|$$

Using the notation:

Range (A,B,C) = R(A,B,C) = the difference between the largest and smallest of A,B,C,

Upper Range (A,B,C) =UR(A,B,C) = the difference between the largest and second largest of A,B,C,

and equation (7) we get the following upper bound for M when n = 3:

$$M \leq 2UR(a_{11},a_{22},a_{33}) + 2R(a_{11},a_{22},a_{23}) + 2R(a_{12},a_{23},a_{31}).$$

The use of the upper and lower bounds on M to simplify the problem of finding MIN M can be shown most directly by considering a specific example.

Let:

$$\mathcal{S} = \begin{bmatrix} 4 & 3 & 3 \\ 7 & 5 & 2 \\ 4 & 1 & 2 \end{bmatrix}$$

Then $a_{i\cdot}$ is one of the set (14,10,7); $a_{\cdot 1}$ = 15 and $a_{\cdot i}$ is one of the set (9,7). All possible pairings of the elements of these sets give the lower bounds for all the values of M. That is, we obtain the lower bounds 2, 6, 10, 12, 14 and 17. Starting with the pairing which gives the smallest member of this group, we calculate the upper bounds on M. For the lower bound $M \geqslant 2$ there are two cases:

$$\mathcal{S}_1 = \begin{bmatrix} 4 & 3 & 3 \\ 7 & 2 & 5 \\ 4 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{S}_2 = \begin{bmatrix} 4 & 1 & 2 \\ 7 & 5 & 2 \\ 4 & 3 & 3 \end{bmatrix}$$

The upper bounds for $\mathcal{S}_1$ and $\mathcal{S}_2$ are 12 and 14 respectively.

Since the upper bound on $\mathcal{S}_1$ is less than or equal to the lower bounds 12, 14 and 17, we may restrict our attention to the six distinct cases which give rise to lower bounds 2, 6 and 10. To decide between the cases remaining required a repeated application of equation (7).

An alternate lower bound for M is:

$$M \geqslant 2R(a_{11},a_{22},a_{33})+2R(a_{12},a_{23},a_{31})-2\mathrm{MIN}\left[UR(a_{11},a_{22},a_{33}),UR(a_{12},a_{23},a_{31})\right]$$

This inequality is easier to apply than equation (7) and may prove helpful in further reducing the set of cases that must be solved exactly by equation (7). In the example above it gives $\mathcal{S}_1 \geqslant 8$ which is the exact value for $\mathcal{S}_1$.

In this example the use of bounds on M reduced the problem by one half. Due to our lack of computational experience we cannot say that this amount of reduction will always occur whenever n = 3. The determination of bounds for the values of M may somewhat simplify the problem

at hand. It is expected, however, that in the majority of cases such a reduction will still leave the problem unmanageable if we must calculate an exact solution for each of the remaining cases. Perhaps the most powerful use of this method is in its application as a test of optimality. For, if we can demonstrate one solution which has its value of M equal to the smallest lower bound we know that solution is an optimal solution.

VII. DISCUSSION.

We have achieved our objective only in a limited sense, in that we have developed solutions for an optimum schedule only in the cases $m = n = 2$ and $m = n = 3$. However, even these limited results are of value, for two reasons. First, as previously indicated, small system sizes of approximately this magnitude are the rule in at least one critical area of fleet operations, Southeast Asia. Second, the work we have done may point the way for further research which might lead to a general solution for any size operation.

Larger problems will include even more complex combinatorial problems than the cases we have studied. An obvious approach towards solving large problems is to develop a computer program. Such an approach has been found practical even for the "one line" classical job scheduling problem when $m > 3$ (1), (2), (3), (5), (7). However, we have avoided involvement with computer solution procedures because the majority of personnel involved with the replenishment scheduling function have neither a multipurpose computer nor a computer specialist available. Moreover, in trying to develop analytical solution techniques we have gained valuable insights into the replenishment operation which would have been lost in the inner workings of a computer had we taken this approach.

One of these insights, in the form of a rather general statement concerning the effects of relatively large operation size on the choice of schedule, follows. As the operation becomes larger in the sense that the number of combatants involved begins to be much greater than the number of supply vessels involved, the planner has more latitude in his choices for optimum sequences. That is, he will be increasingly

less likely that any of the supply ships are not busy. The extent to which this effect occurs is a definite area for further research.

There are several aspects of the actual underway replenishment operation which have not been included in the discussion, other than perhaps in passing up to this point. The first of these is the technique of vertical replenishment. Since this technique is one which may well be a revolutionary factor in replenishment at sea in the future (and having already proven itself a useful adjunct to current operations), it should be mentioned. First, one should consider the question of the method of inclusion of vertical replenishment in the formulation of the problem as presented. Suppose that a particular combatant is capable of receiving all of the supplies it requires from a particular supply vessel by vertical replenishment. If this is the case, we propose that the vector of service times of the combatant in question include a zero as the service time of the combatant at the supply ship in question because the service can be completed while the combatant is either waiting for or completing another service. As a second proposal, we suggest that vertical replenishment be used as a method to make service times more even, that is, to decrease the spread of service times of the various combatants in order to make the system cycle more smoothly. Both of these suggestions are profitable areas for further study.

It is a common practice in underway replenishment to assign one supply vessel to service two combatants at the same time. This is the second of the aspects of the operation which we have not taken into account. One possible method of handling this situation is to split each supply ship into two "effective" servicing units, one to represent the port side, and one the starboard side of the supply ship. Thus one might split a three by six problem into two separate three by three

cases, which can be handled by our methods. The port side of all supply vessels could represent one group, and the starboard side the other. Needless to say, there are many choices and criterion for the assignment of combatants to one group or the other, but, given an assignment, an optimum for the group is obtainable. Note that the spectre of sub-optimization rears its head in the consideration of this suggestion in that the assignment to a specific group must be done in an optimal manner in addition to optimal scheduling within the group, in order to insure an overall optimum. This area is likewise in need of considerable further study.

There exists a need for lifeguard vessels which we have not covered in our analyses to this point. As a safety precaution, some vessel must normally act as this lifeguard vessel for a replenishment station. That means that at any given time there should be m lifeguards on station, one for each of the m supply vessels. The usual procedure is to have some vessel of a smaller, more manuverable type, such as a destroyer, fill this station as it waits to be serviced. However, there is the possibility that there will be no such vessel available to perform the function. There are several methods by which one might handle this consideration if it appears that it will become a serious one. A group of m vessels of the required type may be detailed to act as permanent lifeguards for each of the m supply ships, thus breaking the problem into two parts. The first would be the n - m combatant ships proceeding through the system first, and the second, the m lifeguard ships. If there are enough helicopters available, they might be used as lifeguards. Lastly, if time is of sufficient criticality, one might ignore any problems of this sort which arise on the basis that the time consideration will outweigh the loss in safety. It should become obvious after

a schedule is established and checked just where and when such problems will arise. Decisions should be made at this point as to the most desirable remedy available.

A discussion of some of the more questionable assumptions we have made is now in order. First, and foremost, we have assumed that service times are known quantities. This assumption is obviously contrary to the true state of nature. The time involved has an unknown probability distribution varying generally with the amount of goods required, the type of goods required, with sea state and weather, and possibly many other factors. However, we feel that the determination of this distribution is possible, and is a necessary and important part of the development of optimum schedules in the real world.

We envision the day when tables, or graphs, providing the required service time information will be readily available to the planning agencies. If it can be shown that mean service times are sufficient, then perhaps graphs such as Figure 6 can be set up.

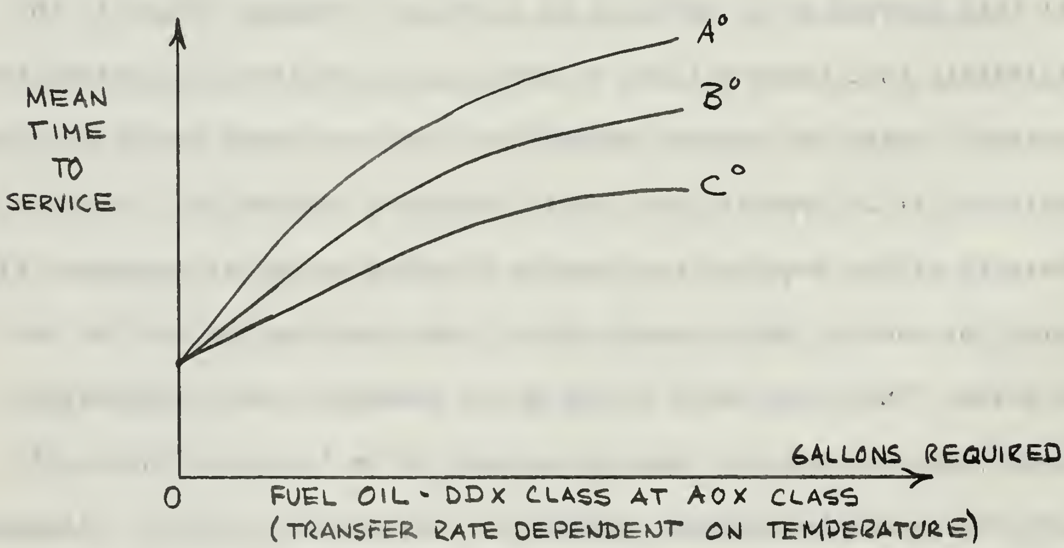

Figure 6

Currently, combatants inform tactical commanders and the replenishment group commander of their requirements for resupply by message, so that the information required by the abcissa is known. We envision a graph or curve of this sort for every class or type of combatant at each type of replenishment vessel. For the newer multipurpose supply ships such as the AOE and AFS classes, the information would have to be of such a nature as to take into account the diversity of supplies carried, as would the information for every type supply vessel except for oilers. The data necessary to build these graphs or curves exists, but considerable effort would be required to sort out, collect, and analyze it.

The assumption was made that the time spent in transit between supply vessels by combatants is both negligible in comparison to service times involved, and identical for all ships. We have shown that if they are identical then transit times cancel out and can be ignored in the actual application of our model. Therefore, the critical assumption is that the transit time is constant. While this is untrue, the variation in these times is generally small enough in comparison with the service times, even for different type combatants, that the error introduced by the assumption will be slight.

It has been stated that the use of computers to solve the problem of scheduling of replenishment at sea is not presently feasible due only to the lack of availability of either computers, or computer trained personnel, or both. As a proposal to alleviate this situation, we suggest the following innovation. The provision of a communications link which would allow the transmission of the required data, such as expected service times, from the tactical planning agency to a shore facility with a readily available computer and computer staff, should be investigated. Both the communications link and the computer service

should be of an on line nature. An arrangement of this sort could provide planning agencies with the necessary answers, and would allow the optimal solution of larger operations than is presently possible using our technique.

VIII. SUMMARY.

In this paper we have investigated the optimal scheduling of underway replenishment operations from the starting point of considering the problem as a job scheduling problem. We have developed general expressions for total waiting time of both combatants and supply vessels, and for the total completion time. Proceeding from this we have developed solution procedures for the optimal schedule under the two criteria Min M and Min T for the cases $m = n = 2$ and $m = n = 3$. Although we have considered the cases individually, the possibility that there is an inductive connection between the first (smaller) cases and larger cases is not ruled out.

The following areas related to the analyses in this paper present some of the most potentially profitable possibilities for further study:

1. Extension of the applicability of analytical solution to a general solution.
2. The extent of increase in latitude for schedule choice for large sized operations.
3. The effect of the introduction of the proposed vertical replenishment modifications upon solutions and the operation as a whole.
4. An investigation of the split supply ship concept.
5. Investigation of the relationship between Min M and Min T solutions.
6. A data collection and analysis effort in order to determine service time distributions.
7. An investigation of the feasibility of the computer data link proposal.

BIBLIOGRAPHY

1. Dudek, Richard A. and Teuton, Otis F.,Development of M-Stage Decision Rule for Scheduling n Jobs through M Machines. Operations Research, vol. 12, number 3, 1964: 471-497.

2. Eastman, W.L., Even, S., and Isaacs, I. M. Bounds for the Optimal Scheduling of n Jobs on m Processors. Management Science, vol. 11, number 2, 1964: 268-279.

3. Lomnicki, Z. A., A "Branch-and-Bound" Algorithm for the Exact Solution of the Three-machine Scheduling Problem. Operational Research Quarterly, vol. 16, number 1, 1965: 89-100.

4. McCullough, D. U., An Application of Queueing Theory to the Operation of Replenishment at Sea. United States Naval Postgraduate School, October, 1966.

5. Palmer, D. S., Sequencing Jobs through a Multi-state Process in the Minimum Total Time-A quick Method of Obtaining a Near Optimum. Operational Research Quarterly, vol. 16, number 1, 1965: 101-107.

6. Pursley, C. R., Lieutenant, USNR. Vertical Replenishment and the UH-46A. U. S. Naval Institute Proceedings, November, 1966: 182-185.

7. Smith, R. D., and Teuton, Otis F., A General Algorithm for the Solution of the n Job, M-Machine Sequencing Problem of the Flow Shop. Operations Research, vol. 15, number 1, 1967: 467-479.

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13. ABSTRACT

The planning for an underway at sea replenishment operation is treated as a special case of the job scheduling problem. Given the service times of the ships to be replenished, general expressions are obtained for total time to complete the operation and total waiting time of the ships involved. Methods to minimize these expressions are developed for several specific cases. Although the largest size of operation considered in this paper is that of three supply ships and three ships to be replenished the methods developed are believed to be extendable to larger operations.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
1. Replenishment at Sea						
2. Underway Replenishment						
3. Scheduling						